2021 FALL MIDTERM EXAMINATION

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1.(15 pt) Consider the linear system over $\mathbb{R}$:

$$\begin{bmatrix} -1 & a & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where a is a constant.

(1) Find all values of a so that the system has a unique solution.
(2) Find all values of a so that the system has infinitely many solutions.
(3) Find all values of a so that the system has no solution.

2.(6 pt) Let W be the subspace of $V = \mathbb{R}^4$ spanned by $(1, 0, -1, 2)$, $(2, 3, 1, 1)$, and $(1, 3, 2, -1)$. Find a basis of the annihilator of W in V^*.

3.(10 pt) Let $a, \lambda_i \in \mathbb{C}$.

(1) Find the characteristic polynomial of

$$\begin{bmatrix} \lambda_1 & a \\ a & \lambda_2 \end{bmatrix}.$$

(2) Find the characteristic polynomial of

$$\begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & -2 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix}.$$

4.(20 pt) Let V, W, U be finite-dimensional $\mathbb{R}$-linear spaces and let $T : V \to W$, $S : W \to U$ be linear maps. Show that $\operatorname{rank}(ST) = \operatorname{rank}(T)$ if and only if ST and T have the same null space.

5.(20 pt) Let V, W be finite-dimensional $\mathbb{R}$-linear spaces and let $T : V \to W$ be a linear map and $T^t : W^* \to V^*$ be the transpose of T. Given $\beta \in W$. Show that the following two statements are equivalent:

(1) $\beta \in \operatorname{Range}(T)$;
(2) for any $f \in \operatorname{Null}(T^t)$, $f(\beta) = 0$.

6.(13 pt) Let V be a finite-dimensional $\mathbb{R}$-linear space and $\alpha \neq \beta \in V$. Show that there exists a linear functional $f \in V^*$ such that $f(\alpha) \neq f(\beta)$.

7.(16 pt) Consider 3 polynomials in $\mathbb{C}[x]$:

$$p_0 = -(x-1)(x+1), \quad p_1 = \frac{1}{2}x(x+1), \quad p_2 = \frac{1}{2}x(x-1).$$

Then a direct check shows that $1 = p_0(x) + p_1(x) + p_2(x)$ and $x = p_1(x) - p_2(x)$. (No need to verify.) Let $T : V \to V$ be a linear map on a finite-dimensional $\mathbb{C}$-linear space V, having minimal polynomial $m_T(x) = x(x-1)(x+1)$. Let $P_i = p_i(T)$. Show that P_0, P_1 and P_2 satisfy the properties

(1) $I = P_0 + P_1 + P_2$;
(2) $T = P_1 - P_2$;
(3) $P_i^2 = P_i$ for all i;
(4) $P_i \cdot P_j = 0$ for $i \neq j$.